GEOMETRY.

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133. Proposed by P. C. CULLEN, Principal of Public Schools, Indianola, Neb.

If the two bisectors, trisectors, quadrasectors, etc., of a triangle are mutually equal, show that the triangle is isosceles.

I. Discussion by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

Eight different "proofs" of this theorem have thus far appeared in THE AMERICAN MATHEMATICAL MONTHLY. It is proposed in this note to point out some fallacious, or otherwise unsatisfactory reasoning in the demonstrations used.

I. Vol. II, No. 5, page 157.

The weak point in this is the apparent assumption that points O and N always fall within the side AB. Now, in the absence of definite knowledge to the contrary, we must assume that point O, for instance, may fall between A and B, on A, or in BA produced. So, either it must be proved that O can fall only between A and B, or else each of the other two cases must be considered, before the truth of the theorem is established.

II. Vol. II, No. 6, page 189.

Aside from the fact that this demonstration is unintelligible to the student merely of geometry, it is quite evident that there exist other relations between x and y than the one selected. Hence, barring the knowledge we may have from other sources, in this method there lurks the suspicion that the hypothesis of equal bisectors may yield also other than isosceles triangles. That is to say, "$x=y$" is not the only conclusion that can be drawn from the premises.

III. Vol. II, No. 6, page 190.

Here the fallacy is in assuming $BD.DC<BE.AE$, simply because $\angle A<\angle C$. With as much show of reason we might say, since $10<12$, and 5, a part of 10, $<9\frac{1}{2}$, a part of 12, therefore $5.5<9\frac{1}{2}.2\frac{1}{2}$.

IV. Vol. II, No. 6, page 160.

A most flagrant fallacy lies on the surface of the statement: "Now the right triangles AEL and ADK are similar, respectively, to AFN and AGM. But these last triangles are equal, and hence the triangles AEL and ADK are equal."

V. Vol. II, No. 6, page 160.

The fallacy here lies in the reasoning that the greater the arc, the greater the chord, which is necessarily true only when the arcs are less than a semi-circumference.

VI. Vol. V, No. 4, page 108.

This is the same "proof" as V above.

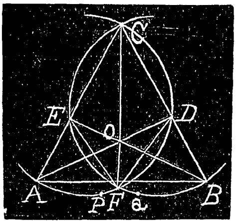
VII. Vol. V, No. 4, page 108.

The last statement in the first paragraph vitiates this demonstration. Because $\angle ACO>\angle ABO$, it does not follow that $AE>AF$; although it is true that if $\angle ACB>\angle ABC$, then $AB>AC$. But in the latter case, we have different criteria than in the former.

The only valid proofs among all the number in the MONTHLY are the following: VI, Vol. II, No. 6, page 189, and III, Vol. V, No. 4, page 109.

II. Demonstration by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let ABC be a triangle such that the bisectors AD and BE, of the angles A and B, respectively, are equal. About the triangles ADC and BEC describe circles. These circles are equal, since the chords AD and BE are equal, by hypothesis, and the arcs subtended by these chords, each being the measure of the angle C.



Therefore the chords BF and DF are equal, since each is subtended by the equal arcs BF and DF, which measure the angle FCB. In like manner, the chords AF and EF are equal. Therefore the triangles AFD and BFE are equal, and hence the perpendicular from F to AD is equal to the perpendicular from F to BD. Therefore OF bisects the angle AOB. Since O is the in-center, the line CO bisects the angle C, and, therefore, CO produced passes through the middle points of P and Q of the equal arcs AFD and BFE, respectively,—points equally distant from AD and BE, and, therefore, on opposite sides of the line OF. Hence the points P and Q must coincide with F, and the line AOF bisects both the angle ACB and AOC. Hence, since AO equals BO, and CO equals CO, and the angle ACO equals the angle BCO, the side AC equals the side BC, and the triangle is isosceles.

The same proof applies for the trisectors, quadrisectors, etc.

NOTE.—The above demonstration is adapted from a demonstration, in the *Educational Times*, London, England, by D. Biddle, editor of the Mathematical Department of that magazine. This seems to me to be a rigorous and direct demonstration and the simplest and most satisfactory that I have ever seen.

Mr. J. S. Mackey says, in the *Educational Times*, "A direct proof of this question will be found in the *London. Edinburgh, and Dublin Philosophical Magazine* (Fourth Series), Vol. XLVII, pages 354-7 (1874)." Mr. R. Tucker, in the same magazine, says, "This question was proposed as question 1907, in the *Lady's and Gentleman's Diary* for 1856, and is solved on page 58 (1857) by Messrs. T. T. Wilkinson, J. W. Elliott (the proposer), and analytically by others. Mr. Wilkinson returns to the problem in his 'Notæ Geometricæ' in the *Diary* for 1859, page 87. An historical note is added on page 88 which traces the question back to the *Nouvelles Annales* for 1842. Professor Sylvester drew attention to the subject in the *Philosophical Magazine* for November, 1852. Dr. Adamson further discusses the matter in the *Philosophical Magazine* for April, May, and June, 1853. The best article I know on question 1907 (*Diary*), appears in section 11 of Wilkinson's 'Notæ Geometricæ' in the *Diary* for 1860, pages 84-86, with a neat proof by Rev. W. Mason. I find that the above references are given in Dr. Mackay's *Euclid*, Page 108. In the key to this work, Dr. Mackay prints a proof by M. Descube."

Mr. W. J. Greenstreet, in the same journal, the *Educational Times*, says, "For this and the similar theorem for two symmedians, see *Intermediarie des Mathématiciens*, Vol. II (1895), pages 151, 325, If the external bisectors of A and B are equal, it does not follow that the triangles are isosceles. The data lead to $4Rr_1 = a^2 + bc$ in the triangle sides a, b, c (V. Mathesis, page 261, 1895)."

This theorem has been published in the MONTHLY three times already in the seven years of the MONTHLY's existence, and has been proposed for publication an innumerable number of times. We have given it considerable attention inasmuch as it has been considered by such noted mathematicians as Professor Sylvester. It is said that Dr. Todhunter tried to find a direct proof of it, but failed. Being the converse of the very simple theorem, *The bisectors of the base angles of an isosceles triangle are equal*, one hastily infers that the proof of it is quite easy. But such is not the case, as is readily seen when a simple and direct proof is attempted, and also from a study of the history of the theorem.

We hope that the above demonstration will be appreciated by our readers, and that in the future efforts, if any, will be made to give simpler proofs than this one.

Demonstrations were again furnished by G. B. M. Zerr, P. C. Cullen, George B. Birkhoff. We also received a few demonstrations that contained fallacies.

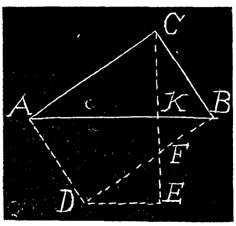
The fallacy in proof V, of Vol. II was also pointed out by Dr. E. S. Loomis of Cleveland, Ohio.

THE PYTHAGOREAN PROPOSITION.

Prof. D. A. Lehman, Baldwin University, Berea, Ohio, offers the following proof of the Pythagorean Proposition:

Given triangle ABC right angled at C, to prove $c^2=a^2+b^2$.

Draw AD parallel to CB, BD parallel to AC, CE perpendicular to AB, (cutting BD at F, and AB at K), HF parallel to CB, and DE parallel to AB.



Of the nine similar triangles thus made, let us consider three: (1) ABC, (2) FBC, (3) DEF.

In (1) and (2), $FB:a=a:b$; $\therefore FB=a^2/b$. Also $CF:c=a:b$; $\therefore CF=ac/b$.

Now $DF=DB-FB=b-a^2/b=\dfrac{b^2-a^2}{b}$.

In (3) and (1), $EF:a=\dfrac{b^2-a^2}{b}:c$; $EF=\dfrac{a}{c}\left(\dfrac{b^2-a^2}{b}\right)$.

$$ABC=\tfrac{1}{2}DBCA=\tfrac{1}{2}ab=\frac{c}{4}CE=\frac{c}{4}[CF+FE]=\frac{c}{4}\left[\frac{a\,c}{b}+\frac{a}{c}\left(\frac{b^2-a^2}{b}\right)\right]$$

$$=\frac{c}{4}\left[\frac{a\,c}{b}+\frac{a\,b}{c}-\frac{a^3}{bc}\right],\ i.\ e.,\ ab=\frac{ac^2}{2b}+\frac{a\,b}{2}-\frac{a^3}{2b},\ \text{or } c^2=a^2+b^2.$$

Dr. E. S. Loomis, Professor of Mathematics in the West Cleveland High School, contributes the following interesting results on Magic Squares:

Suppose in the above figure that the sides AC, BC, and AB are 4, 3, and 5, respectively. On these side construct squares containing 16, 9, and 25 unit squares. In the first row of unit squares along AC put the numbers 16, 6, 5, 19; in the next row above this, put the numbers 11, 13, 14, 8; in the next row above this, put the numbers 15, 9, 10, 12; in the next 4, 18, 17, 7.

These numbers added horizontally, vertically, or diagonally give a sum of 46.

In like manner, place the following numbers as described above along CB beginning at C.

48, 47, 52
53, 49, 45
46, 57, 50

These numbers, when added horizontally, vertically, or diagonally, give a sum=147.

In the same way, place the following numbers in the unit square along AB beginning at A.

15, 16, 33, 30, 31
37, 22, 27, 26, 13
36, 29, 25, 21, 14
18, 24, 23, 28, 32
19, 34, 17, 20, 35

These, when added horizontally, vertically, or diagonally, give a sum=125.

Now $4\times46+3\times147=5\times125$, that is, the sum of all numbers in the hypotenuse—square=the sum of all numbers in the two leg-squares.

134. Proposed by J. C. GREGG, A. M., Superintendent of Schools, Brazil, Ind.

If $ABCD$ is a quadrilateral circumscribing a circle, show that the line joining the middle points of the diagonals AB, CD passes through the center of the circle.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio Univsrsity, Athens, Ohio.

Taking two of the lines as axes and the other two as $lx+my=1$......(1), $l'x+m'y=1$....(2), we find the two vertices of the quadrilateral on the coördinate axes $(0,\ 1/m)$, $(1/l',\ 0)$, and the middle of the diagonal of which these are the extremities $(1/2l',\ 1/2m)$....(3).

The coördinates of the intersection of (1) and (2) are

$$x'=\frac{m-m'}{l'm-lm'},\quad y'=\frac{l-l'}{l'm-lm'}\ldots.(4),$$

which and the origin are the extremities of the second diagonal. The middle of this diagonal is

$$\left(\frac{m-m'}{2(l'm-lm')},\quad \frac{l-l'}{2(l'm-lm')}\right)\ldots.(5).$$

Any conic touching the four lines is of the form

$$(ax+by-1)^2-2\lambda xy=0\ldots.(6).$$

For (6) to be tangent to (1) and (2),

$$\lambda=2(a-l)(b-m)\ldots.(7)\text{ or }\lambda=2(a-l')(b-m')\ldots.(8),$$

respectively. The center of (6) is given by